

The Replica Method and Toda Lattice Equations for QCD₃

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Abstract

We consider the ϵ -regime of QCD in 3 dimensions. It is shown that the leading term of the effective partition function satisfies a set of Toda lattice equations, recursive in the number of flavors. Taking the replica limit of these Toda equations allows us to derive the microscopic spectral correlation functions for the QCD Dirac operator in 3 dimensions. For an even number of flavors we reproduce known results derived using other techniques. In the case of an odd number of flavors the theory has a severe sign problem, and we obtain previously unknown microscopic spectral correlation functions.

1 Introduction

Recently there has been substantial progress in understanding the so-called replica method in the context of the effective low-energy field theory for QCD (QCD_4) [1-7]. While perturbative (series) expansions readily yield correct results with this method [1], it has proved difficult to obtain exact nonperturbative results [8, 9, 10, 11]. A crucial ingredient in formulating a systematic replica method which is reliable also for nonperturbative calculations has been the use of exact relations of the Painlevé [4] or Toda [5, 6, 7] kinds. These relations reflect the fact [12] that the leading-order effective partition function in the so-called ϵ -regime [12]¹ of QCD is what is known as a τ -function of an underlying integrable KP hierarchy [14]. This integrable structure derives back to the spontaneous breaking of chiral symmetry according to the pattern $\text{SU}_L(N_f) \times \text{SU}_R(N_f) \rightarrow \text{SU}(N_f)$, where N_f is the number of light quark flavors. As a consequence of this spontaneous breaking of chiral symmetry and because of the mass gap in QCD the low-energy dynamics of QCD is governed by the Goldstone modes. In the standard counting scheme the low energy theory describing the dynamics of the Goldstone modes is known as chiral perturbation theory, and in this framework the ordinary replica method applies without subtleties [1]. If one instead [12] considers a counting scheme (the ϵ -regime) where the Compton wavelength of the pions is much larger than the linear dimension of the volume, $1/m_\pi \gg L$, then the partition function at leading order reduces to a static integral over the Goldstone manifold. Changing N_f in this integral builds up a series of partition functions which are connected through a Toda lattice equation. This connection between partition functions with different N_f shows how to take the replica limit $N_f \rightarrow 0$. The purpose of this paper is to establish Toda lattice equations for QCD in 3 dimensions (QCD_3) and to show that also these allow us to obtain exact non-perturbative results using the replica method.

QCD_3 cannot undergo spontaneous breaking of chiral symmetry in the usual sense. Moreover, also the notion of gauge field topology is different, a remnant manifestation being the possibility of adding a Chern-Simons term to the action. Nevertheless, a three-dimensional analogue of chiral symmetry breaking (really flavor symmetry breaking, re-interpreted) is possible [15]. The basic idea is that the chiral components of the four-dimensional spinors in three dimensions can be mimicked by two different fermion fields, with masses of equal magnitude but opposite signs. Such mass terms of different signs are possible in an odd number of dimensions, a consequence of the fact the two sets of γ -matrices $\{\gamma_i\}$ and $\{-\gamma_i\}$ form two inequivalent irreducible representations of the Clifford algebra in an odd number of space-time dimensions. The two 2-spinors corresponding to opposite signs of the mass terms may be grouped into one four-spinor of which the top components play the rôle of the left-handed field, while the lower components represent the analogue of the right-handed field in four-dimensional language. Moreover, the associated “chiral” symmetry can break spontaneously, and is *expected* to break spontaneously when the number of fermions species is small. The symmetry breaking pattern is believed to be that of [15, 16]

$$U(2N_f) \rightarrow U(N_f) \times U(N_f). \quad (1.1)$$

There are numerical simulations of QCD_3 on rather small lattices that support this conclusion in the extreme (quenched) case of no dynamical quarks [17]. An odd number of flavors is obviously a more difficult issue. For $2N_f + 1$ flavors of which the $2N_f$ masses are grouped into pairs of equal magnitude but opposite signs it has been argued in ref. [16] that the spontaneous symmetry breaking pattern (1.1) is replaced by

$$U(2N_f + 1) \rightarrow U(N_f + 1) \times U(N_f). \quad (1.2)$$

¹See e.g. ref. [13] for a recent review with an emphasis on the present context.

This case is rather tricky to treat from the effective field theory point of view. An interesting starting point is Random Matrix Theory, which suggests that there are two possible effective field theories, depending on the number of Dirac operator eigenvalues [16]. We will look into this problem in closer detail below. We note that as one further generalization one could also consider the case of $2N_f$ flavors paired with opposite signs, and n unpaired flavors.

The Random Matrix Theory representation of the effective partition function for QCD_3 in the ϵ -regime [16] has lead to a number of intriguing conjectures. In particular, spectral properties of the pertinent 3-dimensional Dirac operator have been argued to be derivable from the joint eigenvalue probability distributions of such a Random Matrix Theory [16, 18], and based on this conjecture universal spectral n -point correlation functions have been found for both an even [19, 20, 21] and an odd [22] number of flavors. First steps toward deriving these results directly from the effective field theory partition function were taken in refs. [23, 3]. In particular, by an impressive series of supermanifold integrations some of the spectral correlation functions were derived from the so-called “supersymmetric” formulation in ref. [23]. In [3] intriguing relations between the QCD_4 and the QCD_3 partition functions in the ϵ -regime were derived. Applying the replica method to these relations a set of identities between the spectral correlation followed. These identities, derived by formally applying the replica limit $N_f \rightarrow 0$, presupposed a meaningful operational way to perform this limit, a non-trivial issue in view of the known subtleties involved [8, 11]. Here, using the replica limit of the Toda lattice equation, we show that these results, and others which are new, can be given a precise meaning.

The new results are for an odd number of flavors. This theory is rather special since after integrating out the fermions the weight inside the partition function is not positive definite. This manifest itself in the eigenvalue density in a number of ways. For example, it is tempting to conclude that eigenvalue density in the theory with an odd number of flavors and an odd number of Dirac operator eigenvalues [16] is an odd function in the chiral limit. However, as we will explain below this is not the case. Although the non-positive weight in QCD_3 with an odd number of flavors comes about quite differently from the sign problem in QCD_4 with a non-zero baryon chemical potential, the lessons learned from QCD_3 may be worth keeping in mind when tackling the sign problem in QCD_4 .

The presentation of this paper is as follows. First we derive the Toda lattice equations for QCD_3 in the ϵ -regime. Then we discuss the extension to bosonic flavors. Given the Toda lattice equations we take the replica limit in section 3 and show how to obtain spectral correlation functions. As in QCD_4 , the Toda lattice equations in QCD_3 also hold for partition functions with both fermions and bosons. This allows us to establish the replica limit of the Toda lattice equation in QCD_3 on the same footing as the supersymmetric method [24]. We show how in section 4. Finally, in section 5 we show how general Toda lattice equations follow from one single Consistency Condition of Random Matrix Theory. We summarize our results in section 6.

2 The Toda Lattice: From QCD_4 to QCD_3

Our starting point is a set of Toda lattice equations which have been derived [25, 26, 14, 6, 7] for the leading term of the effective QCD_4 partition function in the ϵ -regime. Using these and known connections between QCD_4 and QCD_3 in the ϵ -regime we establish the Toda lattice equations for QCD_3 .

We first give our conventions and definitions, which follow ref. [3]. The ϵ -regime is in this (2+1)-dimensional context defined by the large-volume limit where nevertheless $V \ll 1/m_\pi^3$ and m_π generically denotes the pseudo-Goldstone masses of spontaneous breaking of chiral symmetry. In this “ex-

treme” chiral limit where $\mu_i \equiv m_i V \Sigma$ is kept fixed the partition function reduces to a static integral over the Goldstone manifold. The infinite-volume condensate is defined by

$$\Sigma \equiv \lim_{m \rightarrow 0} \lim_{V \rightarrow \infty} |\langle \bar{\psi} \psi \rangle| , \quad (2.1)$$

and m_i are the quark masses.

The leading contribution to the QCD₃ partition function in the ϵ -regime is for an even number $2N_f$ of fermions grouped in pairs with masses of equal magnitude but opposite signs given by [16]

$$\mathcal{Z}_3^{(2N_f)}(\{\mu_i\}) = \int_{U(2N_f)} dU \exp[V \Sigma \text{Re Tr}(\mathcal{M} U \Gamma_5 U^\dagger)] , \quad (2.2)$$

with $\mathcal{M} = \text{diag}(m_1, \dots, m_{N_f}, -m_1, \dots, -m_{N_f})$, $\Gamma_5 \equiv \text{diag}(\mathbf{1}_{N_f}, -\mathbf{1}_{N_f})$ and dU always indicating the Haar measure. This integral can be performed explicitly with the help of the Itzykson-Zuber formula [27], resulting in [20]

$$\mathcal{Z}_3^{(2N_f)}(\{\mu_i\}) = (-1)^{N_f(N_f+1)/2} \frac{\det \begin{pmatrix} \mathbf{A}(\{\mu_i\}) & \mathbf{A}(\{-\mu_i\}) \\ \mathbf{A}(\{-\mu_i\}) & \mathbf{A}(\{\mu_i\}) \end{pmatrix}}{\Delta(\mathcal{M})} , \quad (2.3)$$

where, with our normalization conventions, the $N_f \times N_f$ matrix $\mathbf{A}(\{\mu_i\})$ is defined by

$$\mathbf{A}(\{\mu_i\})_{jl} \equiv (\mu_j)^{l-1} e^{\mu_j} , \quad j, l = 1, \dots, N_f , \quad (2.4)$$

and the Vandermonde determinant is given by

$$\Delta(\mathcal{M}) = \prod_{i>j}^{2N_f} (\mu_i - \mu_j) . \quad (2.5)$$

As mentioned in the Introduction, the case of an odd number of flavors is a bit unusual because the symmetry of the action relies on a paring of masses. Nevertheless, as in the even-flavored theory, the effective partition function has a unique leading-order term in the ϵ -scheme. With $\mathcal{M} = \text{diag}(\{\mu_i\}, \mu, \{-\mu_i\})$ and $\tilde{\Gamma}_5 = (\mathbf{1}_{N_f+1}, -\mathbf{1}_{N_f})$ the partition function becomes [16],

$$\mathcal{Z}_{3+}^{(2N_f+1)}(\mu, \{\mu_i\}) = \int_{U(2N_f+1)} dU \cosh[V \Sigma \text{Re Tr}(\mathcal{M} U \tilde{\Gamma}_5 U^\dagger)] , \quad (2.6)$$

for an even number of Dirac operator eigenvalues, and

$$\mathcal{Z}_{3-}^{(2N_f+1)}(\mu, \{\mu_i\}) = \int_{U(2N_f+1)} dU \sinh[V \Sigma \text{Re Tr}(\mathcal{M} U \tilde{\Gamma}_5 U^\dagger)] , \quad (2.7)$$

for an odd number of Dirac operator eigenvalues. Both of these integrals can again be evaluated explicitly, with the conventions of [3],

$$\mathcal{Z}_{3\pm}^{(2N_f+1)}(\mu, \{\mu_i\}) = (-1)^{N_f(N_f+3)/2} \frac{2^{N_f}}{\Delta(\mathcal{M})} \frac{1}{2} [\det \mathbf{D}(\mu, \{\mu_i\}) \pm (-1)^{N_f} \det \mathbf{D}(-\mu, \{-\mu_i\})] , \quad (2.8)$$

where the $(2N_f + 1) \times (2N_f + 1)$ matrix \mathbf{D} is defined as

$$2^{N_f} \det \mathbf{D}(\mu, \{\mu_i\}) \equiv \det \begin{pmatrix} \mathbf{A}(\mu, \{\mu_i\})_{N_f+1 \times N_f+1} & \mathbf{A}(-\mu, \{-\mu_i\})_{N_f+1 \times N_f} \\ \mathbf{A}(\{-\mu_i\})_{N_f \times N_f+1} & \mathbf{A}(\{\mu_i\})_{N_f \times N_f} \end{pmatrix} , \quad (2.9)$$

and \mathbf{A} is defined in eq. (2.4).

For QCD₄ in the corresponding ϵ -regime the leading contribution to the partition function is²

$$\mathcal{Z}_\nu^{(N_f)}(\{\mu_i\}) = \int_{U(N_f)} dU (\det U)^\nu \exp [V \Sigma \text{ReTr}(\mathcal{M}U)] \quad (2.10)$$

in a sector of topological charge ν [12, 28]. The quark mass matrix in 4 dimensions is $\mathcal{M}=\text{diag}(m_1, \dots, m_{N_f})$. Also this integral can be explicitly evaluated for integer ν [29]:

$$\mathcal{Z}_\nu^{(N_f)}(\{\mu_i\}) = \frac{\det \mathbf{B}(\{\mu_i\})}{\Delta(\{\mu_i^2\})}, \quad (2.11)$$

where the matrix \mathbf{B} in eq. (2.11) is given by

$$\mathbf{B}(\{\mu_i\})_{jl} = \mu_j^{l-1} I_\nu^{(l-1)}(\mu_j) , \quad j, l = 1, \dots, N_f , \quad (2.12)$$

with $I_\nu^{(l)}$ the l 'th derivative modified Bessel function I_ν , and the denominator is given by the Vandermonde determinant of, in this case, squared quark masses,

$$\Delta(\{\mu_i^2\}) \equiv \prod_{i>j}^{N_f} (\mu_i^2 - \mu_j^2) = \det_{i,j} \left[(\mu_i^2)^{j-1} \right] . \quad (2.13)$$

We use on purpose the same notation μ_i in both QCD₃ and QCD₄, since relations between these two widely different theories exist when these dimensionless parameters are identified in the two theories. The QCD₄ partition function for non-integer ν is defined by analytical continuation in the index ν of the Bessel functions [3]. We stress at this point that the explicit formulas (2.3), (2.8) and (2.11) of course are only valid for N_f taking positive integer values. Below we shall return to the issue of how to deal with cases where N_f is zero, or even negative.

2.1 Two theorems

Our main results will be based on the following two relations between the QCD₃ and QCD₄ partition functions. These relations were proven in ref. [3] and hold when the normalization conventions are as stated above.

THEOREM I

$$\mathcal{Z}_3^{(2N_f)}(\{\mu_i\}) = \pi^{N_f} \mathcal{Z}_{\nu=-1/2}^{(N_f)}(\{\mu_i\}) \mathcal{Z}_{\nu=+1/2}^{(N_f)}(\{\mu_i\}) . \quad (2.14)$$

THEOREM II

$$\mathcal{Z}_{3^\pm}^{(2N_f+1)}(\mu, \{\mu_i\}) = \pi^{N_f} \sqrt{\frac{\pi\mu}{2}} \mathcal{Z}_{\nu=\mp1/2}^{(N_f+1)}(\mu, \{\mu_i\}) \mathcal{Z}_{\nu=\pm1/2}^{(N_f)}(\{\mu_i\}) . \quad (2.15)$$

We have written these theorems explicitly for real and positive masses $\{\mu_i\}$. When taking discontinuities in the complex plane one should always recall that only the absolute value enters here. In what follows we will for notational simplicity normally not explicitly display the associated factors that equal unity for real and positive values (usually terms of the form $x/\sqrt{x^2}$).

²We use a notation in which we do not explicitly indicate that this is the 4-dimensional partition function; the partition function is seen to be the one for QCD₄ by the labeling according to topological charge ν . There should be no source of confusion as we will never explicitly consider the case $\nu = 3$.

2.2 Toda lattice equations for QCD₃

The two theorems (2.14) and (2.15) can nicely be combined to derive Toda lattice equations for QCD₃ on the basis of the known Toda equations for QCD₄. As a first illustration of this procedure, consider the case of N_f degenerate flavors of mass x in QCD₄. In this case the leading term of the QCD₄ partition function satisfies the differential equation [25]

$$(x\partial_x)^2 \ln \mathcal{Z}_\nu^{(N_f)}(x) = 2N_f x^2 \frac{\mathcal{Z}_\nu^{(N_f+1)}(x)\mathcal{Z}_\nu^{(N_f-1)}(x)}{\left[\mathcal{Z}_\nu^{(N_f)}(x)\right]^2}. \quad (2.16)$$

We can turn this into a Toda equation for QCD₃ by use of Theorem I and II. By considering the case of an even number of flavors $2N_f$ with degenerate masses x we get

$$(x\partial_x)^2 \ln \mathcal{Z}_3^{(2N_f)}(x) = 4N_f x \frac{\mathcal{Z}_{3^+}^{(2N_f+1)}(x)\mathcal{Z}_{3^-}^{(2N_f-1)}(x) + \mathcal{Z}_{3^-}^{(2N_f+1)}(x)\mathcal{Z}_{3^+}^{(2N_f-1)}(x)}{\left[\mathcal{Z}_3^{(2N_f)}(x)\right]^2}. \quad (2.17)$$

This is the QCD₃ Toda lattice equation analogous to the QCD₄ equation (2.16). We note the quite generic feature of the QCD₃ equation having two contributions on the right hand side, originating from having two different partition functions with an odd number of flavors. Of course, we might have derived the Toda equation (2.17) directly from the explicit formulas (2.3) and (2.8). But the present shortcut through use of Theorems I and II is obviously a much simpler route.

We next consider the QCD₄ Toda lattice equations with N_f degenerate flavors of mass x and n degenerate flavors of mass y . These are [25]

$$x\partial_x(x\partial_x + y\partial_y) \ln \mathcal{Z}_\nu^{(N_f,n)}(x,y) = 2N_f x^2 \frac{\mathcal{Z}_\nu^{(N_f+1,n)}(x,y)\mathcal{Z}_\nu^{(N_f-1,n)}(x,y)}{\left[\mathcal{Z}_\nu^{(N_f,n)}(x,y)\right]^2}, \quad (2.18)$$

and [6]

$$(x\partial_x y\partial_y) \ln \mathcal{Z}_\nu^{(N_f,n)}(x,y) = 4N_f n x^2 y^2 \frac{\mathcal{Z}_\nu^{(N_f+1,n+1)}(x,y)\mathcal{Z}_\nu^{(N_f-1,n-1)}(x,y)}{\left[\mathcal{Z}_\nu^{(N_f,n)}(x,y)\right]^2}. \quad (2.19)$$

The corresponding Toda lattice equations for the QCD₃ partition functions are readily obtained from the theorems. To keep it simple, we will start by looking at $2N_f$ flavors with degenerate masses $\pm x$ and $2n$ flavors with degenerate masses $\pm y$ (both with the usual pairing). After re-expressing the Toda equations in terms of QCD₃ partition functions we get

$$\begin{aligned} & x\partial_x(x\partial_x + y\partial_y) \ln \mathcal{Z}_3^{(2N_f,2n)}(x,y) \\ = & 4N_f x \frac{\mathcal{Z}_{3^+}^{(2N_f+1,2n)}(x,y)\mathcal{Z}_{3^-}^{(2N_f-1,2n)}(x,y) + \mathcal{Z}_{3^-}^{(2N_f+1,2n)}(x,y)\mathcal{Z}_{3^+}^{(2N_f-1,2n)}(x,y)}{\left[\mathcal{Z}_3^{(2N_f,2n)}(x,y)\right]^2}. \end{aligned} \quad (2.20)$$

Once again the right hand side has two contributions. The generalization to the case where the $2n$ masses are paired but non-degenerate is obtained simply by replacing y with $\{y_i\}$ and $(x\partial_x + y\partial_y)$ by $(x\partial_x + \sum_{i=1}^n y_i \partial_{y_i})$. Following the same route as described above, we find that the QCD₃ equivalent

of (2.19) for an even number of flavors is

$$\begin{aligned} & x\partial_x y \partial_y \log \mathcal{Z}_3^{(2N_f, 2n)}(x, y) \\ = & 4N_f n x^3 y \left(\frac{\mathcal{Z}_{3^-}^{(2N_f+2, 2n+1)}(x, y) \mathcal{Z}_{3^+}^{(2N_f-2, 2n-1)}(x, y)}{\mathcal{Z}_{3^+}^{(2N_f+1, 2n)}(x, y) \mathcal{Z}_{3^-}^{(2N_f-1, 2n)}(x, y)} + \frac{\mathcal{Z}_{3^+}^{(2N_f+2, 2n+1)}(x, y) \mathcal{Z}_{3^-}^{(2N_f-2, 2n-1)}(x, y)}{\mathcal{Z}_{3^-}^{(2N_f+1, 2n)}(x, y) \mathcal{Z}_{3^+}^{(2N_f-1, 2n)}(x, y)} \right). \end{aligned} \quad (2.21)$$

Despite appearances, the right hand side is symmetric under interchange of x and y .

We now turn to the partition functions with an odd number of flavors. Starting at the simplest case with $2N_f$ flavors of paired masses $\pm x$ and one additional flavor of mass x we get from (2.16):

$$\begin{aligned} & (x\partial_x)^2 \ln \mathcal{Z}_{3^\pm}^{(2N_f+1)}(x) \\ = & 4x \left((N_f + 1) \frac{\mathcal{Z}_{3^\pm}^{(2N_f+3)}(x) \mathcal{Z}_{3^\mp}^{(2N_f+1)}(x)}{\left[\mathcal{Z}_3^{(2N_f+2)}(x) \right]^2} + N_f \frac{\mathcal{Z}_{3^\mp}^{(2N_f+1)}(x) \mathcal{Z}_{3^\pm}^{(2N_f-1)}(x)}{\left[\mathcal{Z}_3^{(2N_f)}(x) \right]^2} \right). \end{aligned} \quad (2.22)$$

For the QCD₃ version of (2.18) where we have $2N_f$ paired and degenerate masses $\pm x$ and $2n + 1$ flavors of mass y we find

$$\begin{aligned} & x\partial_x(x\partial_x + y\partial_y) \ln \mathcal{Z}_{3^\pm}^{(2N_f, 2n+1)}(x, y) \\ = & 4N_f x \left(\frac{\mathcal{Z}_{3^\pm}^{(2N_f+1, 2n+2)}(x, y) \mathcal{Z}_{3^\mp}^{(2N_f-1, 2n+2)}(x, y)}{\left[\mathcal{Z}_3^{(2N_f, 2n+2)}(x, y) \right]^2} + \frac{\mathcal{Z}_{3^\mp}^{(2N_f+1, 2n)}(x, y) \mathcal{Z}_{3^\pm}^{(2N_f-1, 2n)}(x, y)}{\left[\mathcal{Z}_3^{(2N_f, 2n)}(x, y) \right]^2} \right). \end{aligned} \quad (2.23)$$

As the final Toda lattice equation for QCD₃ we also give the analogue of (2.22),

$$\begin{aligned} & x\partial_x y \partial_y \log \mathcal{Z}_{3^\pm}^{(2N_f, 2n+1)}(x, y) \\ = & 4N_f x^3 y \left((n + 1) \frac{\mathcal{Z}_{3^\pm}^{(2N_f+2, 2n+3)}(x, y) \mathcal{Z}_{3^\mp}^{(2N_f-2, 2n+1)}(x, y)}{\mathcal{Z}_{3^\mp}^{(2N_f+1, 2n+2)}(x, y) \mathcal{Z}_{3^\pm}^{(2N_f-1, 2n+2)}(x, y)} \right. \\ & \left. + n \frac{\mathcal{Z}_{3^\mp}^{(2N_f+2, 2n+1)}(x, y) \mathcal{Z}_{3^\pm}^{(2N_f-2, 2n-1)}(x, y)}{\mathcal{Z}_{3^\pm}^{(2N_f+1, 2n)}(x, y) \mathcal{Z}_{3^\mp}^{(2N_f-1, 2n)}(x, y)} \right), \end{aligned} \quad (2.24)$$

valid for an odd number of flavors.

2.3 Extension to bosonic flavors

Before proceeding to use the replica method we must ascertain that the Theorems I-II are consistent with what has been understood about the QCD₄ partition functions for both zero and a negative number of flavors. We should first clarify what is meant by this. For zero flavors the partition functions are “quenched”, the fermion determinants are entirely absent, and there can be no mass dependence. It is clearly convenient to choose the normalization so that in this case the partition function simply equals unity. A negative number of fermionic flavors is defined by raising the determinant to the corresponding negative number. This is equivalent to a partition function of the same number of complex fields with bosonic statistics (and thus violating the spin-statistics theorem, but these bosons are never considered as external physical states). On the QCD₄ side the generalization of the partition function formula (2.11) was given in ref. [30, 5, 31] for an arbitrary number N_f and N_b of fermionic

and bosonic species, respectively. For *positive* N_f and N_b the explicit expression reads, in a hopefully obvious notation,

$$\mathcal{Z}_\nu^{(N_f|N_b)}(\{x_f\}|\{y_b\}) = \frac{\det[z_i^{j-1} \mathcal{J}_{\nu+j-1}(z_i)]_{i,j=1,\dots,N_f+N_b}}{\prod_{j>i=1}^{N_f} (x_j^2 - x_i^2) \prod_{j>i=1}^{N_b} (y_j^2 - y_i^2)}, \quad (2.25)$$

where $z_i = x_i$ for $i = 1, \dots, N_f$, $z_{N_f+i} = y_i$ for $i = 1, \dots, N_b$, $\mathcal{J}_{\nu+j-1}(z_i) \equiv I_{\nu+j-1}(x_i)$ for $i = 1, \dots, N_f$, and $\mathcal{J}_{\nu+j-1}(z_{N_f+i}) \equiv (-1)^{j-1} K_{\nu+j-1}(y_i)$ for $i = 1, \dots, N_b$. The generalization to negative integers is done through identifications such as $\mathcal{Z}_\nu^{(-N_f|N_b)} = \mathcal{Z}_\nu^{(0|N_f+N_b)}$ and so on. Thus, while the $N_f = 1$ QCD₄ partition function is given by $\mathcal{Z}_\nu^{(1)}(x) = I_\nu(x)$, the $N_f = -1$ partition function reads $\mathcal{Z}_\nu^{(-1)}(x) = K_\nu(x)$, where $K_\nu(x)$ is the modified Bessel function. We therefore know how to define the right hand side of the two theorems for a negative number of flavors. The question is whether we can define the left hand side for zero or a negative number of flavors in QCD₃ in this way. Setting $N_f = 0$ in eq. (2.14) gives $\mathcal{Z}_3^{(0)} = 1$ from Theorem I, *i.e.*, with the correct normalization. Having gained some faith in this procedure, we can infer the QCD₃ partition function for $N_f = -1$ from Theorem II:

$$\mathcal{Z}_{3^\pm}^{(-1)}(x) = \sqrt{\frac{x}{2\pi}} K_{\pm 1/2}(x) = e^{-x}. \quad (2.26)$$

This agrees exactly with what we should expect on general principle: For $N_f = -1$ (one boson) there is no spontaneous symmetry breaking at all, and the leading term of the partition function is simply, by a generalization of the argument by Leutwyler and Smilga in the 4-dimensional case [28], the exponentiation of the leading term free energy $F = mV\Sigma = \mu$. It thus appears that we can continue with the indicated identifications, and Theorem I will then give us the partition function for $N_f = -2$:

$$\mathcal{Z}_3^{(-2)}(x) = \frac{1}{\pi} K_{-1/2}(x) K_{1/2}(x). \quad (2.27)$$

This procedure obviously continues for higher (negative) values of N_f .

We can formalize the above arguments by making the following conjectures regarding supersymmetric generalizations of Theorems I and II:

$$\mathcal{Z}_3^{(2N_f|2N_b)}(\{x_i\}|\{y_i\}) = \pi^{N_f-N_b} \mathcal{Z}_{-1/2}^{(N_f|N_b)}(\{x_i\}|\{y_i\}) \mathcal{Z}_{1/2}^{(N_f|N_b)}(\{x_i\}|\{y_i\}) \quad (2.28)$$

$$\mathcal{Z}_{3^\pm}^{(2N_f+1|2N_b)}(\{x_i\}, x|\{y_i\}) = \pi^{N_f-N_b} \sqrt{\frac{\pi x}{2}} \mathcal{Z}_{\mp 1/2}^{(N_f+1|N_b)}(\{x_i\}, x|\{y_i\}) \mathcal{Z}_{\pm 1/2}^{(N_f|N_b)}(\{x_i\}|\{y_i\}) \quad (2.29)$$

$$\mathcal{Z}_{3^\pm}^{(2N_f|2N_b+1)}(\{x_i\}|\{y_i\}, y) = \pi^{N_f-N_b} \sqrt{\frac{y}{2\pi}} \mathcal{Z}_{\pm 1/2}^{(N_f|N_b+1)}(\{x_i\}|\{y_i\}, y) \mathcal{Z}_{\mp 1/2}^{(N_f|N_b)}(\{x_i\}|\{y_i\}). \quad (2.30)$$

In all cases we have been able to check these identities are consistent with what we know from other sources. We note in particular that if they are correct we have circumvented the quite difficult task of evaluating all the so-called Efetov-Wegner terms in the supersymmetric version of the effective Lagrangian [23]. It appears that the most direct way to prove these identities may be through an explicit evaluation of the related supersymmetric Random Matrix Theory integral, as in ref. [31].

3 Replica limit of the QCD₃ Toda lattice equations

As an introduction let us consider QCD₄ with N_f quarks with masses $\{\mu_f\}$ and n fermions of mass x . Using the replica method the partially quenched chiral condensate, the resolvent, is then defined as

$$G_\nu^{(N_f)}(x, \{\mu_f\}) \equiv \lim_{n \rightarrow 0} \frac{1}{n} \frac{\partial}{\partial x} \ln \mathcal{Z}_\nu^{(N_f, n)}(\{\mu_f\}, x). \quad (3.31)$$

The spectral density of the Dirac operator in QCD₄ with N_f flavors is given by

$$\rho_\nu^{(N_f)}(\lambda, \{\mu_f\}) \equiv \left\langle \sum_k \delta(\lambda_k - \lambda) \right\rangle_{N_f}, \quad (3.32)$$

where $\langle \dots \rangle_{N_f}$ is the vacuum expectation value in QCD₄ with N_f flavors. The delta functions can be expressed as the discontinuity of the resolvent across the imaginary axis

$$\begin{aligned} \rho_\nu^{(N_f)}(\lambda, \{\mu_f\}) &= \frac{1}{2\pi} \text{Disc } G_\nu^{(N_f)}(x, \{\mu_f\}) \Big|_{x=i\lambda} \\ &= \frac{1}{2\pi} \lim_{\varepsilon \rightarrow 0} [G_\nu^{(N_f)}(i\lambda + \varepsilon) - G_\nu^{(N_f)}(i\lambda - \varepsilon)]. \end{aligned} \quad (3.33)$$

The resolvent and the density in QCD₃ with an even number of flavors $2N_f$ follow analogously [3]. In the replica formulation the resolvent is defined from the partition functions with $2N_f$ flavors with paired masses $\{\pm\mu_f\}$ as well as $2n$ replica flavors with paired masses $\pm x$:

$$G_3^{(2N_f)}(x, \{\mu_f\}) = \lim_{n \rightarrow 0} \frac{1}{2n} \frac{\partial}{\partial x} \ln \mathcal{Z}_3^{(2N_f, 2n)}(\{\mu_f\}, x). \quad (3.34)$$

As for the partition functions we use a similar notation as in QCD₄, the two are easily separated by the explicit topological index ν .

The eigenvalue density is defined as in (3.32) except that the average is in QCD₃ with $2N_f$ flavors. However, because the replicated flavors have paired masses the discontinuity over the imaginary axis in the complex quark mass plane of the resolvent becomes

$$\begin{aligned} \frac{1}{2\pi} \text{Disc } G_3^{(2N_f)}(x, \{\mu_f\}) \Big|_{x=i\lambda} &= \frac{1}{4\pi} \lim_{\varepsilon \rightarrow 0} \left\langle \sum_k \frac{2\varepsilon}{(\lambda_k + \lambda)^2 + \varepsilon^2} + \frac{2\varepsilon}{(\lambda_k - \lambda)^2 + \varepsilon^2} \right\rangle_{2N_f} \\ &= \frac{1}{2} (\rho_3^{(2N_f)}(-\lambda, \{\mu_f\}) + \rho_3^{(2N_f)}(\lambda, \{\mu_f\})). \end{aligned} \quad (3.35)$$

The limit $n \rightarrow 0$ in the defining equations for the resolvents (3.31) and (3.34) must obviously be taken with care, since the partition functions entering the right hand side are only known for integer values of n . For more than two decades it was widely believed [8] that one could at best obtain small or large argument expansions of the true result using the replica method. Recent developments, see *e.g.* [9, 10], attempted to go beyond this, but also that approach was met with some criticism [11]. However, with [4, 5, 6, 7] this situation has drastically changed. One tool to perform the replica limit correctly is the Toda lattice equations for the leading-order QCD₃ partition functions. How it works in detail is perhaps best explained by working out a couple of examples.

Let us start with the simplest case in QCD₃ namely the fully quenched spectral density of the Dirac operator. In order to obtain this we first determine the fully quenched resolvent from the Toda lattice equation (2.17)

$$\begin{aligned} \lim_{n \rightarrow 0} \frac{1}{2n} (x \partial_x)^2 \log \mathcal{Z}_3^{(2n)}(x) &= x \partial_x x G_3^{(0)}(x) \\ &= 2x (\mathcal{Z}_{3+}^{(1)}(x) \mathcal{Z}_{3-}^{(-1)}(x) + \mathcal{Z}_{3-}^{(1)}(x) \mathcal{Z}_{3+}^{(-1)}(x)), \end{aligned} \quad (3.36)$$

where we have used that $\mathcal{Z}_3^{(N_f=0)} = 1$. The partition functions can conveniently be found from (2.15) which then gives

$$\partial_x x G_3^{(0)}(x) = x K_{1/2}(x) (I_{-1/2}(x) + I_{1/2}(x)) = 1. \quad (3.37)$$

The solution with boundary condition $G_3^{(0)}(x)$ finite in the limit $x \rightarrow 0$ follows readily,

$$G_3^{(0)}(x) = 1 . \quad (3.38)$$

Taking the discontinuity of the resolvent we get

$$\frac{1}{2} \left(\rho_3^{(0)}(-\lambda) + \rho_3^{(0)}(\lambda) \right) = \rho_3^{(0)}(\lambda) = \frac{1}{\pi}, \quad (3.39)$$

where we have used that the even-flavored quenched spectral density is an even function in λ . Eq. (3.39) is just the result derived from Random Matrix Theory [16], where, due to the absence of the determinant, it simply corresponds to the bulk microscopic UE density, which is flat.

Next, we work out the eigenvalue density in QCD₃ with an even number of flavors. Using the same technical steps as in [5] the resolvent in QCD₃ with an even number of flavors follows from the Toda lattice equation (2.20). Consider the easiest example $N_f = 1$ which gives

$$\lim_{n \rightarrow 0} \frac{1}{2n} x \partial_x (x \partial_x + y \partial_y) \ln \mathcal{Z}_3^{(2n,2)}(x, y) = 2x \frac{\mathcal{Z}_{3+}^{(1,2)}(x, y) \mathcal{Z}_{3-}^{(2,-1)}(y, x) + \mathcal{Z}_{3-}^{(1,2)}(x, y) \mathcal{Z}_{3+}^{(2,-1)}(y, x)}{\left[\mathcal{Z}_3^{(2)}(y) \right]^2}, \quad (3.40)$$

where once again the partition functions on the right hand side can be found from (2.14) and (2.15). This can be integrated to give a resolvent

$$G_3^{(2)}(x, y) = \frac{x^2 - y^2 - x + ye^{-x}(\coth(y)\sinh(x) + \cosh(x)\tanh(y))}{x^2 - y^2} \quad (3.41)$$

and from this the density in QCD₃ with two paired masses follows:

$$\rho_3^{(2)}(\lambda, y) = \rho_3^{(2)}(-\lambda, y) = \frac{1}{\pi} - \frac{y(\cos^2 \lambda \tanh y + \sin^2 \lambda \coth y)}{\pi(\lambda^2 + y^2)}. \quad (3.42)$$

This is in complete agreement with the result found in Random Matrix Theory [20].

Now we move on to the odd sector. We look at (2.24) with one real flavor and $2n$ replicas for both an even and odd number of eigenvalues. Taking the replica limit of this Toda lattice equation gives us

$$\lim_{n \rightarrow 0} \frac{1}{2n} x \partial_x (x \partial_x + y \partial_y) \ln \mathcal{Z}_{3\pm}^{(2n,1)}(x, y) = 2x \left(\frac{\mathcal{Z}_{3\pm}^{(1,2)}(x, y) \mathcal{Z}_{3\mp}^{(2,-1)}(y, x) + \mathcal{Z}_{3\mp}^{(1)}(x) \mathcal{Z}_{3\pm}^{(-1)}(x)}{\left[\mathcal{Z}_3^{(2)}(y) \right]^2} \right). \quad (3.43)$$

We start by looking at the case of an even number of eigenvalues. In this case we get the following resolvent

$$G_{3+}^{(1)}(x, y) = \frac{x^2 - y^2 + e^{-x}(y \cosh(x) \tanh(y) - x \sinh(x))}{x^2 - y^2}, \quad (3.44)$$

which then gives the symmetric part of the eigenvalue density in QCD₃ with one flavor and an even number of eigenvalues

$$\frac{1}{2} \left(\rho_{3+}^{(1)}(-\lambda, y) + \rho_{3+}^{(1)}(\lambda, y) \right) = \frac{1}{\pi} - \frac{\lambda \cos \lambda \sin \lambda + y \cos^2 \lambda \tanh y}{\pi(\lambda^2 + y^2)}. \quad (3.45)$$

In the massless case, $y = 0$, this agrees with the prediction from Random Matrix Theory [22] and thus gives indirect confirmation of the conjectured form of the supersymmetric partition functions in QCD₃.

We next consider the case of an odd number of Dirac operator eigenvalues, for which microscopic spectral correlation functions have not been evaluated previously due to the unusual behavior of the partition function. This behavior does not obstruct our replica approach, in fact the evaluation of the spectral density is completely analogous to that in the even sector. The replica limit of the Toda lattice equation in this case gives the resolvent

$$G_{3^-}^{(1)}(x, y) = \frac{x^2 - y^2 + e^{-x}(y \sinh(x) \coth(y) - x \cosh(x))}{x^2 - y^2}, \quad (3.46)$$

which then leads to the even part of the density in QCD₃ with one flavor and an odd number of eigenvalues

$$\frac{1}{2} (\rho_{3^-}^{(1)}(-\lambda, y) + \rho_{3^-}^{(1)}(\lambda, y)) = \frac{1}{\pi} + \frac{\lambda \cos \lambda \sin \lambda - y \sin^2 \lambda \coth y}{\pi(\lambda^2 + y^2)}. \quad (3.47)$$

This new result is plotted in figures 1 and 2, the plots showing respectively the massless case ($y = 0$) and a massive case ($y = 10$). Note in particular that in the massless case the sum does not vanish: the eigenvalue density in the massless case is *not* an odd function of λ . Let us clarify this point. It is tempting to assume that the spectral density of the Dirac operator eigenvalues will be odd in λ . Our explicit calculation above shows that it is not the case, but one can gain a better understanding of this phenomenon if one is willing to use the Random Matrix Theory representation for the eigenvalue density [16]. To this end, let Z denote the partition function in the Random Matrix Theory (only in the microscopic limit does this theory agree with the static field integral [16]),

$$Z_{3,N}^{(N_f)}(\{m_f\}) = \int \prod_{k=1}^N d\lambda_k \prod_{k < l=1}^N |\lambda_l - \lambda_k|^2 \prod_{k=1}^N e^{-NV(\lambda_k^2)} \prod_{f=1}^{N_f} (\lambda_k + im_f) \quad (3.48)$$

for a suitable potential $V(\lambda^2)$ [19]. We will use this representation to highlight the special properties of the density in the theory with both N_f and N odd. It is sufficient to consider $N_f = 1$. First note that the partition function in this theory goes to zero linearly with m . This in itself is not alarming, and for example also the effective QCD₄ partition function (2.11) vanishes like m^ν in the limit $m \rightarrow 0$. However in that case the analogous Random Matrix Theory includes an explicit factor m^ν in front of the eigenvalue representation, and this is easily canceled out when calculating the spectral density from the matrix integral. The small-mass behavior of the integral (3.48) is more complicated, as the mass m mixes with the eigenvalues. To be precise, we simply *define* the spectral density by the expectation value (3.32). From eq. (3.48) it then follows that the eigenvalue density is given by

$$\rho_{3,N}^{(1)}(\lambda_1, m) = \frac{(\lambda_1 + im)e^{-NV(\lambda_1^2)} \int \prod_{k=2}^N d\lambda_k \prod_{k < l=1}^N |\lambda_l - \lambda_k|^2 \prod_{k=2}^N e^{-NV(\lambda_k^2)} (\lambda_k + im)}{Z_{3,N}^{(1)}(m)}. \quad (3.49)$$

For odd N and $\lambda_1 \neq 0$ the eigenvalue density diverges like $1/m$ for $m \rightarrow 0$. However, for $\lambda_1 = 0$ the $m \rightarrow 0$ limit is finite. Using the eigenvalue representation above it is also easy to show that

$$\rho_{3,N}^{(1)}(-\lambda, m) = \rho_{3,N}^{(1)}(\lambda, -m) \quad (3.50)$$

and that the sum

$$\rho_{3,N}^{(1)}(\lambda, m) + \rho_{3,N}^{(1)}(-\lambda, m)$$

appearing in (3.47), is finite for all values of λ even in the limit $m \rightarrow 0$. The divergence as $m \rightarrow 0$ thus resides entirely in the odd part of the eigenvalue density, and this odd part is not probed by the discontinuity of the resolvent.

Our results in the odd- N_f theory with an odd number of Dirac operator eigenvalues therefore seem to be well understood also from the Random Matrix Theory representation. As one additional non-trivial check on our results, we note that the eigenvalue density (3.47) satisfies the correct decoupling condition of approaching the fully quenched spectral density when the mass y goes to infinity.

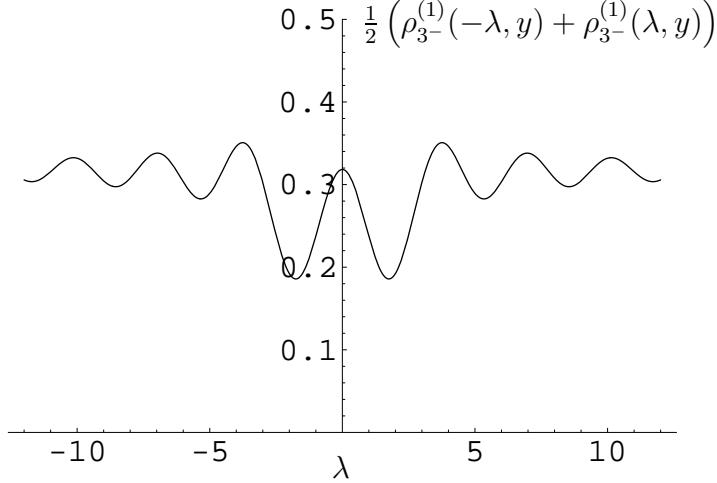


Figure 1: Plot of the symmetric part of the microscopic spectral density in QCD₃ for one massless quark and an odd number of Dirac eigenvalues. The spectral density takes the value $1/\pi$ at $\lambda = 0$.

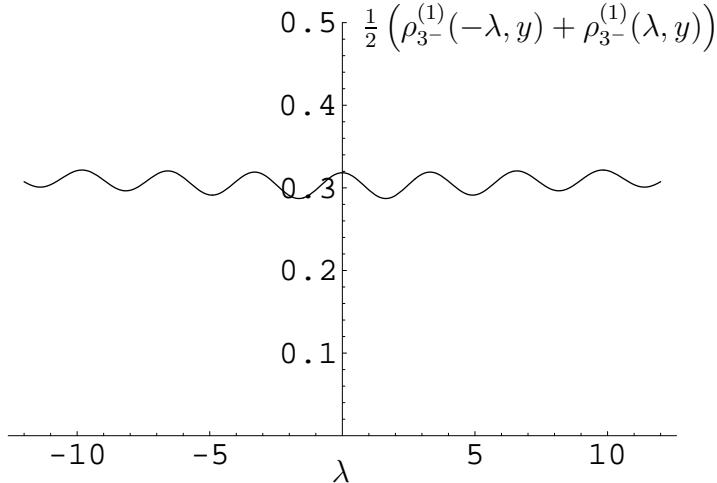


Figure 2: Plot of the symmetric part of the spectral density for one relatively heavy quark, $y = 10$, and an odd number of Dirac eigenvalues. The amplitude of the oscillations is smaller than in the massless case. As the mass, y , goes to infinity the amplitude decrease as $1/y$, and the density converges toward the quenched spectral density, $1/\pi$, as expected.

So far we have given several examples of how the eigenvalue density can be derived from the replica limit of the Toda lattice equation. One can also derive the chiral susceptibility in this fashion. This is in fact much simpler since the Toda lattice equation already has a double derivative, and no integration

is therefore required.

To determine the chiral susceptibility

$$\chi_3^{(0)}(x, y) = \lim_{n \rightarrow 0, m \rightarrow 0} \frac{1}{4nm} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \ln \mathcal{Z}_3^{(2m, 2n)}(x, y), \quad (3.51)$$

we take the replica limit of the Toda lattice equation, (2.22), and find

$$\chi_3^{(0)}(x, y) = 4x^2 \left(\frac{\mathcal{Z}_{3-}^{(2,1)}(x, y) \mathcal{Z}_{3+}^{(-2,-1)}(x, y)}{\mathcal{Z}_{3+}^{(1)}(x) \mathcal{Z}_{3-}^{(-1)}(x)} + \frac{\mathcal{Z}_{3+}^{(2,1)}(x, y) \mathcal{Z}_{3-}^{(-2,-1)}(x, y)}{\mathcal{Z}_{3-}^{(1)}(x) \mathcal{Z}_{3+}^{(-1)}(x)} \right). \quad (3.52)$$

Using Theorem II and (2.30) we find

$$\chi_3^{(0)}(x, y) = \frac{e^{-x-y} \sinh(x+y)}{(x+y)^2}. \quad (3.53)$$

The double discontinuity of this function gives the double symmetric combination of the two point function [3]

$$\rho_3^{(0)}(\lambda_1, \lambda_2) + \rho_3^{(0)}(-\lambda_1, \lambda_2) + \rho_3^{(0)}(\lambda_1, -\lambda_2) + \rho_3^{(0)}(-\lambda_1, -\lambda_2) = \frac{1}{(2\pi)^2} \text{Disc } \chi_3^{(0)}(x, y) \Big|_{x=i\lambda_1, y=i\lambda_2} \quad (3.54)$$

We note that all of these cases, and any other we can consider, give the detailed support for the general relations that formally can be derived on the basis of Theorems I and II and the replica definitions (3.31) and (3.34) alone [3]. These are, for the one-point functions,

$$\rho_3^{(2N_f)}(\lambda; \{\mu_i\}) = \frac{1}{2} \left[\rho_{\text{QCD}_4}^{(N_f, \nu=+1/2)}(\lambda; \{\mu_i\}) + \rho_{\text{QCD}_4}^{(N_f, \nu=-1/2)}(\lambda; \{\mu_i\}) \right] \quad (3.55)$$

for an even number of flavors, and

$$\rho_{3+}^{(2N_f+1)}(\lambda; \{\mu_i\}, \mu) + \rho_{3+}^{(2N_f+1)}(-\lambda; \{\mu_i\}, \mu) = \rho_{\nu=-1/2}^{(N_f+1)}(\lambda; \{\mu_i\}, \mu) + \rho_{\nu=+1/2}^{(N_f)}(\lambda; \{\mu_i\}), \quad (3.56)$$

for an odd number of flavors in the “+”-sector.³

In ref. [3] the case of an odd number of flavors in the “-”-sector was not considered, but we can easily derive the analogous relation by use of Theorem II. We find

$$\rho_{3-}^{(2N_f+1)}(\lambda; \{\mu_i\}, \mu) + \rho_{3-}^{(2N_f+1)}(-\lambda; \{\mu_i\}, \mu) = \rho_{\nu=+1/2}^{(N_f+1)}(\lambda; \{\mu_i\}, \mu) + \rho_{\nu=-1/2}^{(N_f)}(\lambda; \{\mu_i\}), \quad (3.57)$$

for which our (3.47) indeed is a special case.

We also remind that similar general relations can be worked out for higher k -point spectral correlation functions [3], and that Toda lattice equations can give the justification for the formal replica manipulations that lead to these relations. Because of the pairing of masses with different signs these relations do not just involve the k -point spectral correlators themselves, but combinations with different signs (see ref. [3] for details). We have already seen examples of this phenomenon for the spectral 1-point functions derived above. Also the chiral susceptibility (3.53) gives the symmetric combination of the spectral 2-point function. To isolate the spectral correlation functions directly one needs to combine the replica method with more than just the discontinuity formula of the mass-paired resolvent and higher-order versions thereof.

³Note that ref. [3] incorrectly assumed that $\rho_{3+}^{(2N_f+1)}(\lambda; \{\mu_i\}, \mu)$ is an even function of λ also in the case of $\mu \neq 0$.

4 Graded Toda lattice equation

While the Toda lattice equation allows us to take the replica limit in a well defined and correct way it does not fully explain why the analytic continuation in the number of flavors works. However, it is not difficult to understand why the replica limit of the Toda lattice equation and the supersymmetric [24] method give the same answer. To make this transparent [7] we consider supersymmetric versions of the Toda lattice equations and quench these as in the supersymmetric method [24]. Referring to the graded symmetry such Toda lattice equations have been called graded Toda lattice equations [7].

Two graded Toda lattice equations have been found for QCD₄ [7]. Here we will take a closer look at one of these namely the supersymmetric variant of (2.18) with N_f degenerate fermionic flavors of paired masses $\pm x$ and N_b degenerate bosonic flavors of paired masses $\pm y$

$$x\partial_x(x\partial_x + y\partial_y) \ln \mathcal{Z}_\nu^{(N_f|N_b)}(x|y) = 2N_f x^2 \frac{\mathcal{Z}_\nu^{(N_f+1|N_b)}(x|y)\mathcal{Z}_\nu^{(N_f-1|N_b)}(x|y)}{\left[\mathcal{Z}_\nu^{(N_f|N_b)}(x|y)\right]^2}. \quad (4.58)$$

The generalized versions of Theorems I and II can be used to translate this equation in to a graded Toda lattice equation for QCD₃

$$\begin{aligned} & x\partial_x(x\partial_x + y\partial_y) \ln \mathcal{Z}_3^{(2N_f|2N_b)}(x|y) \\ &= 4N_f x \frac{\mathcal{Z}_{3+}^{(2N_f+1|2N_b)}(x|y)\mathcal{Z}_{3-}^{(2N_f-1|2N_b)}(x|y) + \mathcal{Z}_{3-}^{(2N_f+1|2N_b)}(x|y)\mathcal{Z}_{3+}^{(2N_f-1|2N_b)}(x|y)}{\left[\mathcal{Z}_3^{(2N_f|2N_b)}(x|y)\right]^2}. \end{aligned} \quad (4.59)$$

It suffices to look at the special case of $N_f = 1$ and $N_b = 1$, and quench it supersymmetrically. We start by focusing on the lhs

$$\lim_{y \rightarrow x} \frac{1}{2} x\partial_x(x\partial_x + y\partial_y) \ln \mathcal{Z}_3^{(2|2)}(x, -x|y, -y) = x\partial_x x G_3^{(0)}(x), \quad (4.60)$$

where we have used the supersymmetric definition of the fully quenched resolvent

$$G_3^{(0)}(x) \equiv \lim_{y \rightarrow x} \partial_x \ln \mathcal{Z}_3^{(2|2)}(x, -x|y, -y). \quad (4.61)$$

Taking $y \rightarrow x$ on the rhs of (4.59) with $N_f = N_b = 1$ thus leads to

$$x\partial_x x G_3^{(0)}(x) = 2x \left(\mathcal{Z}_{3+}^{(1)}(x)\mathcal{Z}_{3-}^{(-1)}(x) + \mathcal{Z}_{3-}^{(1)}(x)\mathcal{Z}_{3+}^{(-1)}(x) \right), \quad (4.62)$$

in complete consistency with (3.37) derived from the replica limit of the Toda lattice equation. This immediate consistency comes about since the supersymmetric generating functionals satisfy exactly the same Toda lattice equation as the fermionic and bosonic hierarchy of partition functions.

5 Consistency Conditions and Toda Lattice Equations

It is interesting to note that the Toda equations for QCD₄ are special cases of a more general relation which was derived in ref. [14], and there referred to as “Consistency Condition II” due to this equation’s

origin as a consistency condition in Random Matrix Theory [21]. This more general relation reads, after rotating to real masses,

$$\begin{aligned}\mathcal{Z}_\nu^{(N_f+2)}(x, y, \{\mu_f\}) &= \frac{1}{(x^2 - y^2)\mathcal{Z}_\nu^{(N_f)}(\{\mu_f\})} \\ &\times \left[\mathcal{Z}_\nu^{(N_f+1)}(y, \{\mu_f\}) \left(\sum_{f=1}^{N_f} \mu_f \partial_{\mu_f} + x \partial_x \right) \mathcal{Z}_\nu^{(N_f+1)}(x, \{\mu_f\}) - (x \leftrightarrow y) \right].\end{aligned}\quad (5.63)$$

Taking the completely mass-degenerate case $x = y = \mu_f$ for all f , and switching to the conventional normalization for the mass-degenerate case, we recover the Toda lattice eq. (2.17). Similarly, if we take $x = y = \mu_f$ for $f = 1, \dots, N_f$ and $\mu_f = z$ for $f = N_f + 1, \dots, N_f + n$ we recover the Toda eq. (2.18). It appears that the single equation (5.63) is the generator of all Toda lattice equations.

In ref. [21] it was stated that an analogous consistency condition exists for the QCD_3 case, but no details were given. Here we supply this missing equation, which turns out to be of a quite different kind. We begin by noting that the kernel $K(x, y, \{\mu_f\})$ associated with the Random Matrix Theory (3.48) can be written in terms of the even-flavored effective QCD_3 partition function [21]

$$K(x, y, \{\mu_i\}) = \prod_f^{N_f} \sqrt{(x^2 + \mu_f^2)(y^2 + \mu_f^2)} \frac{\mathcal{Z}_3^{(2N_f+2)}(ix, iy, \{\mu_i\})}{\mathcal{Z}_3^{(2N_f)}(\{\mu_i\})}, \quad (5.64)$$

while in general from Random Matrix Theory it is known that it also can be represented in terms of the associated orthogonal polynomials $P_n(x, \{\mu_i\})$ (see *e.g.*, ref. [19] for details),

$$K(x, y, \{\mu_i\}) = \prod_f^{N_f} \sqrt{(x^2 + \mu_f^2)(y^2 + \mu_f^2)} \frac{P_{2N-1}(x, \{\mu_i\})P_{2N}(y, \{\mu_i\}) - P_{2N}(x, \{\mu_i\})P_{2N-1}(y, \{\mu_i\})}{x - y}. \quad (5.65)$$

For this particular Random Matrix Theory ensemble the orthogonal polynomials split into two disjoint sectors of odd and even order. Now, since also these orthogonal polynomials in the large- N limit can be expressed directly in terms of the finite-volume partition functions [21],

$$\begin{aligned}P_{2N}(x, \{\mu_i\}) &= \frac{\mathcal{Z}_{3+}^{(2N_f+1)}(ix, \{\mu_i\})}{\mathcal{Z}_3^{(2N_f)}(\{\mu_i\})} \\ P_{2N+1}(x, \{\mu_i\}) &= \frac{\mathcal{Z}_{3-}^{(2N_f+1)}(ix, \{\mu_i\})}{\mathcal{Z}_3^{(2N_f)}(\{\mu_i\})}\end{aligned}\quad (5.66)$$

we can combine eqs. (5.64)-(5.66) into a consistency condition that must be satisfied by the effective partition functions. In contrast to the QCD_4 case [21] we do not need to expand to first non-trivial order in $1/N$, and the relation will therefore be *algebraic*. We find, after rotating into real masses throughout and fixing the normalization constant,

$$\frac{\mathcal{Z}_3^{(2N_f+2)}(x, y, \{\mu_i\})}{x - y} = \frac{2}{\mathcal{Z}_3^{(2N_f)}(\{\mu_i\})} \frac{\mathcal{Z}_{3-}^{(2N_f+1)}(x, \{\mu_i\})\mathcal{Z}_{3+}^{(2N_f+1)}(y, \{\mu_i\}) - \mathcal{Z}_{3-}^{(2N_f+1)}(x, \{\mu_i\})\mathcal{Z}_{3+}^{(2N_f+1)}(y, \{\mu_i\})}{\mathcal{Z}_3^{(2N_f)}(\{\mu_i\})}. \quad (5.67)$$

The three main types of partition functions that enter QCD_3 are thus not independent.

A particularly useful special case of eq. (5.67) is that of $y = -x$. Using $\mathcal{Z}_{3-}^{(2N_f+1)}(-x, \{\mu_i\}) = -\mathcal{Z}_{3-}^{(2N_f+1)}(x, \{\mu_i\})$ as well as $\mathcal{Z}_{3+}^{(2N_f+1)}(-x, \{\mu_i\}) = \mathcal{Z}_{3+}^{(2N_f+1)}(x, \{\mu_i\})$ we get a simple factorization identity,

$$\mathcal{Z}_3^{(2N_f+2)}(x, -x, \{\mu_i\}) = \frac{2 \mathcal{Z}_{3+}^{(2N_f+1)}(x, \{\mu_i\}) \mathcal{Z}_{3-}^{(2N_f+1)}(x, \{\mu_i\})}{x \mathcal{Z}_3^{(2N_f)}(\{\mu_i\})}. \quad (5.68)$$

This special case actually also follows from combining the relations of Theorems I-II, and re-expressing results in terms of QCD₃ partition functions alone. However, the more general relation (5.67) seems to go beyond what is contained in these theorems.

The relation (5.68) gives rise to non-trivial identities among spectral correlation functions if we combine it with the replica method in the same way we derived the general relations (3.55)-(3.57). We can illustrate this very easily by focusing on the spectral density itself. To this end, consider the special case where we add $2N$ fermions of paired (common up to a sign) masses $\pm\mu$ to the $2N_f$ paired masses μ_i . The resolvents of the different theories can then be related:

$$\begin{aligned} G_3^{(2N_f+2)}(x, -x, \{\mu_i\}, \mu) &= \lim_{N \rightarrow 0} \frac{1}{2N} \partial_\mu \ln \mathcal{Z}_3^{(2N_f+2N+2)}(x, -x, \{\mu_i\}, \mu) \\ &= \lim_{N \rightarrow 0} \frac{1}{2N} \left[\partial_\mu \ln \mathcal{Z}_{3+}^{(2N_f+2N+1)}(x, \{\mu_i\}, \mu) + \partial_\mu \ln \mathcal{Z}_{3-}^{(2N_f+2N+1)}(x, \{\mu_i\}, \mu) \right. \\ &\quad \left. - \partial_\mu \ln \mathcal{Z}_3^{(2N_f+2N)}(\{\mu_i\}, \mu) \right] \\ &= G_{3+}^{(2N_f+1)}(x, \{\mu_i\}, \mu) + G_{3-}^{(2N_f+1)}(x, \{\mu_i\}, \mu) - G_3^{(2N_f)}(\{\mu_i\}, \mu) \end{aligned} \quad (5.69)$$

which in turn implies an identity among the spectral densities,

$$2 \left[\rho_3^{(2N_f+2)}(\lambda, x, -x, \{\mu_i\}) + \rho_3^{(2N_f)}(\lambda, \{\mu_i\}) \right] = \rho_{3+}^{(2N_f+1)}(\lambda, x, \{\mu_i\}) + \rho_{3+}^{(2N_f+1)}(-\lambda, x, \{\mu_i\}) \\ + \rho_{3-}^{(2N_f+1)}(\lambda, x, \{\mu_i\}) + \rho_{3-}^{(2N_f+1)}(-\lambda, x, \{\mu_i\}). \quad (5.70)$$

Needless to say, this general relation also follows from combining eqs. (3.55)-(3.57).

6 Conclusions

The purpose of this paper was to demonstrate that the replica method based on Toda lattice equations successfully can be applied to QCD₃ in the ϵ -regime. The result is a series of exact statements about spectral correlation functions for the Dirac operator of that theory in the phases where spontaneous symmetry breaking occurs. The Toda lattice equations were derived on the basis of known Toda equations for the effective QCD₄ partition functions in the ϵ -regime and exact relations between the two theories. Our results give the detailed support for general relations that formally can be derived by the replica method applied directly on the relations between QCD₃ and QCD₄. Because our corresponding Random Matrix Theory ensemble incorporate a number of determinants, these results go much beyond, but include as a special case, what has already been shown for the ordinary unitary ensemble.

Armed with a reliable tool for computing spectral correlation functions by means of the replica method, we have also attacked a case that previously has not been considered in detail in the literature: an odd number of flavors in a regularization with an odd number of Dirac operator eigenvalues. We have shown that the peculiarities surrounding this case do not lead to pathologies in the spectral properties of the theory. This can also be understood from the point of view of the corresponding Random Matrix Theory once the singularity at vanishing quark mass has been treated carefully.

As in 4-dimensional QCD, the replica method based on Toda lattice equations allows for a smooth connection to the supersymmetric approach. In particular we have conjectured the form of the supersymmetric generalization of the relationship between the effective QCD_3 and QCD_4 partition functions in the extreme limit of the ϵ -regime. With the corresponding 4-dimensional expressions known in closed analytical form this gives us a very compact expression for the analogous expression in QCD_3 , an expression which will be extremely tedious and difficult to derive directly in the chiral Lagrangian form (previous results in that direction do not include the so-called Efetov-Wegner “boundary” terms [23]). We suspect that a simple proof may be obtained by reverting to the Random Matrix Theory expression for the partition function.

Finally we have illustrated how Toda lattice equations for QCD_4 follow as special cases of a more general expression derived originally from the Random Matrix Theory representation. Interestingly, when we extend the proof in the most straightforward manner to QCD_3 we do not recover our Toda lattice equations, but instead an algebraic relationship between the involved effective partition functions. Again, as a special case this relation can be reduced to an equation that can also be derived on the basis of the connection between QCD_3 and QCD_4 . When applying the replica method to this relation, now justified on account of the Toda approach, we obtain exact relations between the spectral correlation functions involved. We have verified explicitly in simple cases that this general constraint indeed is satisfied.

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